ON STABLE SOLUTIONS OF BIHARMONIC PROBLEM WITH POLYNOMIAL GROWTH

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Abstract. We prove the nonexistence of smooth stable solution to the biharmonic problem $\Delta^2 u = u^p$, $u > 0$ in $\mathbb{R}^N$ for $1 < p < \infty$ and $N < 2(1+x_0)$, where x_0 is the largest root of the following equation:

$$x^4 - \frac{32p(p+1)}{(p-1)^2}x^2 + \frac{32p(p+1)(p+3)}{(p-1)^3}x - \frac{64p(p+1)^2}{(p-1)^4} = 0.$$

In particular, as $x_0 > 5$ when $p > 1$, we obtain the nonexistence of smooth stable solution for any $N \le 12$ and $p > 1$. Moreover, we consider also the corresponding problem in the half space $\mathbb{R}^N_+$, or the elliptic problem $\Delta^2 u = \lambda(u+1)^p$ on a bounded smooth domain Ω with the Navier boundary conditions. We will prove the regularity of the extremal solution in lower dimensions. Our results improve the previous works in [20, 19, 2, 4].

1. Introduction

Consider the biharmonic equation

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^N \tag{1.1}$$

where $N \ge 5$ and $p > 1$. Let

$$\Lambda(\phi) := \int_{\mathbb{R}^N} |\Delta\phi|^2 dx - p\int_{\mathbb{R}^N} u^{p-1}\phi^2 dx, \quad \forall\, \phi \in H^2(\mathbb{R}^N). \tag{1.2}$$

A solution u is said stable if $\Lambda(\phi) \ge 0$ for any test function $\phi \in H^2(\mathbb{R}^N)$.

In this note, we prove the following classification result.

Theorem 1.1. *Let $N \ge 5$ and $p > 1$. The equation* (1.1) *has no classical stable solution, if $N < 2 + 2x_0$ with x_0 the largest root of the polynomial*

$$H(x) = x^4 - \frac{32p(p+1)}{(p-1)^2}x^2 + \frac{32p(p+1)(p+3)}{(p-1)^3}x - \frac{64p(p+1)^2}{(p-1)^4}. \tag{1.3}$$

Moreover, we have $x_0 > 5$ for any $p > 1$. Consequently, if $N \le 12$, (1.1) *has no classical stable solution for all $p > 1$.*

For the corresponding second order problem:

$$\Delta u + |u|^{p-1}u = 0 \text{ in } \mathbb{R}^N, \quad p > 1. \tag{1.4}$$

Farina has obtained the optimal Liouville type result for all finite Morse index solutions. He proved in [7] that a smooth finite Morse index solution to (1.4) exists, if and only if $p \ge p_{JL}$ and $N \ge 11$, or $p = \frac{N+2}{N-2}$ and $N \ge 3$. Here p_{JL} is the so-called Joseph-Lundgren exponent, see [12].

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The nonexistence of positive solutions to (1.1) are showed if $p < \frac{N+4}{N-4}$, and all entire solutions are classified if $p = \frac{N+4}{N-4}$, see [15, 18]. On the other hand, the radially symmetric solutions to (1.1) are studied in [8, 9, 13, 14]. In particular, Karageorgis proved that the radial entire solution to (1.1) is stable if and only if $p \geq p_{JL_4}$ and $N \geq 13$. Here p_{JL_4} stands for the corresponding Joseph-Lundgren exponent to Δ^2, see [14].

The general fourth order case (1.1) is more delicate, since the integration by parts argument used by Farina cannot be adapted easily. The first nonexistence result for general stable solution was proved by Wei and Ye in [20], they proposed to consider (1.1) as a system

$$-\Delta u = v, \ -\Delta v = u^p \ \text{ in } \mathbb{R}^N, \tag{1.5}$$

and introduced the idea to use different test functions with u but also v. Using estimates in [17] they showed that for $N \leq 8$, (1.1) has no smooth stable solutions. For $N \geq 9$, using blow-up argument, they proved that the classification holds still for $p < \frac{N}{N-8} + \epsilon_N$ with $\epsilon_N > 0$, but without any explicit value of ϵ_N. This result was improved by Wei, Xu & Yang in [19] for $N \geq 20$ with a more explicit bound.

Very recently, using the stability for system (1.5) and some interesting iteration argument, Cowan proved that, see Theorem 2 in [2], there is no smooth stable solution to (1.1), if $N < 2 + \frac{4(p+1)}{p-1}t_0$, where

$$t_0 = \sqrt{\frac{2p}{p+1}} + \sqrt{\frac{2p}{p+1} - \sqrt{\frac{2p}{p+1}}}, \quad \forall\, p > 1. \tag{1.6}$$

In particular, if $N \leq 10$, (1.1) has no stable solution for any $p > 1$.

However, the study for radial solutions in [14] suggests the following conjecture:

A smooth stable solution to (1.1) *exists if and only if* $p \geq p_{JL_4}$ *and* $N \geq 13$.

Consequently, the Liouville type result for stable solutions of (1.1) should hold true for $N \leq 12$ with any $p > 1$, that's what we prove here. More precisely, by Theorem 1 in [14], the radial entire solutions to (1.1) are unstable if and only if

$$\frac{N^2(N-4)^2}{16} < pQ_4\left(-\frac{4}{p-1}\right), \quad \text{where } Q_4(m) = m(m-2)(m+N-2)(m+N-4). \tag{1.7}$$

The l.h.s. comes from the best constant of the Hardy-Rellich inequality (see [16]): Let $N \geq 5$,

$$\int_{\mathbb{R}^N} |\Delta\varphi|^2 dx \geq \frac{N^2(N-4)^2}{16}\int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^4}dx, \quad \forall\, \varphi \in H^2(\mathbb{R}^N).$$

The r.h.s of (1.7) comes from the weak radial solution $w(x) = |x|^{-\frac{4}{p-1}}$. When $p > \frac{N+4}{N-4}$, we can check that $w \in H^2_{loc}(\mathbb{R}^N)$ and

$$\Delta^2 w = Q_4\left(-\frac{4}{p-1}\right)w^p \ \text{ in } \mathcal{D}'(\mathbb{R}^N).$$

As $w^{p-1}(x) = |x|^{-4}$, using the Hardy-Rellich inequality, the condition (1.7) means just that w is not a stable solution in $\mathbb{R}^N$, i.e.

$$\exists\, \varphi \in H^2(\mathbb{R}^N) \text{ such that } \Lambda_w(\varphi) := \int_{\mathbb{R}^N} |\Delta\varphi|^2 dx - p\int_{\mathbb{R}^N} Q_4\left(-\frac{4}{p-1}\right) w^{p-1}\varphi^2 dx < 0.$$

If we denote $N = 2 + 2x$, a direct calculation shows that (1.7) is equivalent to $H_{JL_4}(x) < 0$, where

$$H_{JL_4}(x) := (x^2-1)^2 - \frac{32p(p+1)}{(p-1)^2}x^2 + \frac{32p(p+1)(p+3)}{(p-1)^3}x - \frac{64p(p+1)^2}{(p-1)^4}.$$

By [9], (1.7) is equivalent to $N < 2+2x_1$ if x_1 denotes the largest root of H_{JL_4}. We can remark the nearness between the polynomial H in Theorem 1.1 and H_{JL_4}, since $H(x) - H_{JL_4}(x) = 2x^2 - 1$.

Furthermore, Theorem 1.1 improves the bound given in [2] for all $p > 1$. Indeed, there holds $x_0 > \frac{2(p+1)}{p-1}t_0$, see Lemmas 2.2 and 2.4 below.

Recall that to handle the equation (1.1), we prove in general that $v = -\Delta u > 0$ in $\mathbb{R}^N$ using average function on the sphere, see [18]. Applying the blow up argument as in [17, 20], we can assume then u and v are uniformly bounded in $\mathbb{R}^N$. Therefore the following Souplet's estimate in [17] holds true in $\mathbb{R}^N$, which was established for any *bounded* solution u of (1.1):

$$v \geq \sqrt{\frac{2}{p+1}}u^{\frac{p+1}{2}}. \tag{1.8}$$

Here we propose a new approach. Without assuming the boundedness of u or showing immediately the positivity of v, we prove first some integral estimates for stable solutions of (1.1), which will enable us the estimate (1.8). This idea permits us to handle more general biharmonic equations: Let $N \geq 5$ and $p > 1$, consider

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \Sigma \subset \mathbb{R}^N, \quad u = \Delta u = 0 \text{ on } \partial\Sigma. \tag{1.9}$$

Let $E = H^2(\Sigma) \cap H_0^1(\Sigma)$ and

$$\Lambda_0(\phi) := \int_\Sigma |\Delta\phi|^2 dx - p\int_\Sigma u^{p-1}\phi^2 dx, \quad \forall\, \phi \in E. \tag{1.10}$$

A solution u of (1.9) is said stable if $\Lambda_0(\phi) \geq 0$ for any $\phi \in E$.

Proposition 1.2. *Let u be a classical stable solution of* (1.9) *with $\Sigma = \mathbb{R}^N$, or the half space $\Sigma = \mathbb{R}^N_+$, or the exterior domain $\Sigma = \mathbb{R}^N \setminus \overline{\Omega}, \mathbb{R}^N_+ \setminus \overline{\Omega}$ where Ω is a bounded smooth domain of $\mathbb{R}^N$. Then the inequality* (1.8) *holds in Σ, consequently $v > 0$ in Σ.*

Using this, we obtain the Liouville type result for (1.9) on half space situation, which improves the result in [20] for wider range of N, and without assuming the boundedness of u or $v = -\Delta u$.

Theorem 1.3. *Let x_0 be defined in Theorem 1.1. If $N < 2+2x_0$, there exists no classical stable solution of* (1.9) *if $\Sigma = \mathbb{R}^N_+$.*

Our proof combines also many ideas coming from the previous works [20, 4, 2]. Briefly, for (1.1), we use different test functions to both equations of the system (1.5) and we make use of the following inequality in [4] (see also [2, 5]): If u is a stable solution of (1.1), then

$$\int_{\mathbb{R}^N} \sqrt{p}u^{\frac{p-1}{2}}\varphi^2 dx \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2 dx, \quad \forall\, \varphi \in C_0^1(\mathbb{R}^N). \tag{1.11}$$

This will enable us two estimates. By suitable combination, we prove that for any stable solution u to (1.1), $\phi \in C_0^2(\mathbb{R}^N)$ and $s \geq 1$, there holds

$$L(s) < 0 \quad \Rightarrow \int_{\mathbb{R}^N} u^p v^{s-1} \phi^2 dx \leq C \int_{\mathbb{R}^N} v^s \left(|\Delta(\phi^2)| + |\nabla \phi|^2\right) dx \tag{1.12}$$

Here L is a polynomial of degree 4, see (2.9) below, and the constant C depends only on p and s. Applying then the iteration argument of Cowan in [2], we show that $u \equiv 0$ if $N < 2 + 2x_0$, which is a contradiction, since u is positive.

Using similar ideas, we consider the elliptic equation on bounded domains:

$$(P_\lambda) \qquad \begin{cases} \Delta^2 u = \lambda(u+1)^p & \text{in a bounded smooth domain } \Omega \subset \mathbb{R}^N, \quad N \geq 1 \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known (see [1, 10]) that there exists a critical value $\lambda^* > 0$ depending on $p > 1$ and Ω such that

- If $\lambda \in (0, \lambda^*)$, (P_λ) has a minimal and classical solution u_λ which is stable;
- If $\lambda = \lambda^*$, $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$ is a weak solution to (P_{λ^*}), u^* is called the *extremal solution*.
- No solution of (P_λ) exists whenever $\lambda > \lambda^*$.

In [3, 20], it was proved that if $1 < p < \left(\frac{N-8}{N}\right)_+^{-1}$ or equally when $N < \frac{8p}{p-1}$, the extremal solution u^* is smooth. Recently, Cowan & Ghoussoub improve the above result by showing that u^* is smooth if $N < 2 + \frac{4(p+1)}{p-1} t_0$ with t_0 in (1.6), so u^* is smooth for any $p > 1$ when $N \leq 10$. Our result is

Theorem 1.4. *The extremal solution u^* is smooth if $N < 2 + 2x_0$ with x_0 given by Theorem 1.1. In particular, u^* is smooth for any $p > 1$ if $N \leq 12$.*

Remark that our proof does not use the *a priori* estimate of $v = -\Delta u$ as in [3, 4].

The paper is organized as follows. We prove some preliminary results and Proposition 1.2 in section 2. The proofs of Theorems 1.1, 1.3 and 1.4 are given respectively in section 3 and 4.

2. Preliminaries

We show first how to obtain the estimate (1.8) for stable solutions of (1.9). Our idea is to use the stability condition (1.10) to get some decay estimate for stable solutions of (1.9). In the following, we denote by B_r the ball of center 0 and radius $r > 0$.

Lemma 2.1. *Let u be a stable solution to* (1.9) *and $v = -\Delta u$, there holds*

$$\int_{\Sigma \cap B_R} \left(v^2 + u^{p+1}\right) dx \leq C R^{N-4-\frac{8}{p-1}}, \quad \forall\, R > 0. \tag{2.1}$$

Proof. We proceed similarly as in Step 1 of the proof for Theorem 1.1 in [20], but we do not assume here that $v > 0$ or u is bounded in Σ. For any $\xi \in C^4(\Sigma)$ verifying $\xi = \Delta\xi = 0$ on $\partial\Sigma$ and $\eta \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} \int_\Sigma (\Delta^2 \xi) \xi \eta^2 dx = & \int_\Sigma [\Delta(\xi\eta)]^2 dx + \int_\Sigma \left[-4(\nabla\xi \cdot \nabla\eta)^2 + 2\xi\Delta\xi|\nabla\eta|^2\right] dx \\ & + \int_\Sigma \xi^2 \Big[2\nabla(\Delta\eta) \cdot \nabla\eta + (\Delta\eta)^2\Big] dx. \end{aligned} \tag{2.2}$$

The proof is direct as for Lemma 2.3 in [20], noticing just that in the integrations by parts, all boundary integration terms on $\partial\Sigma$ vanish under the Navier conditions for ξ.

Take $\xi = u$, a solution of (1.9) into (2.2), there holds

$$\begin{aligned}&\int_\Sigma [\Delta(u\eta)]^2 dx - \int_\Sigma u^{p+1}\eta^2 dx\\ &= 4\int_\Sigma (\nabla u \nabla \eta)^2 dx + 2\int_\Sigma uv|\nabla\eta|^2 dx - \int_\Sigma u^2\Big[2\nabla(\Delta\eta)\cdot\nabla\eta + (\Delta\eta)^2\Big]dx\end{aligned}$$

where $v = -\Delta u$. Using $\phi = u\eta$ in (1.10), we obtain easily

$$\begin{aligned}&\int_\Sigma \Big[(\Delta(u\eta))^2 + u^{p+1}\eta^2\Big]dx\\ &\le C_1\int_\Sigma \Big[|\nabla u|^2|\nabla\eta|^2 + u^2|\nabla(\Delta\eta)\cdot\nabla\eta| + u^2(\Delta\eta)^2\Big]dx + C_2\int_\Sigma uv|\nabla\eta|^2 dx.\end{aligned}\tag{2.3}$$

Here and in the following, C or C_i denotes generic positive constants independent on u, which could be changed from one line to another. As $\Delta(u\eta) = 2\nabla u\cdot\nabla\eta + u\Delta\eta - v\eta$, from (2.3), we get

$$\begin{aligned}&\int_\Sigma \left[v^2\eta^2 + u^{p+1}\eta^2\right]dx\\ &\le C_1\int_\Sigma \Big[|\nabla u|^2|\nabla\eta|^2 + u^2|\nabla(\Delta\eta)\cdot\nabla\eta| + u^2(\Delta\eta)^2\Big]dx + C_2\int_\Sigma uv|\nabla\eta|^2 dx.\end{aligned}\tag{2.4}$$

On the other hand, as $u = 0$ on $\partial\Sigma$,

$$\begin{aligned}2\int_\Sigma |\nabla u|^2|\nabla\eta|^2 dx &= \int_\Sigma \Delta(u^2)|\nabla\eta|^2 dx + 2\int_\Sigma uv|\nabla\eta|^2 dx\\ &= \int_\Sigma u^2\Delta(|\nabla\eta|^2)dx + 2\int_\Sigma uv|\nabla\eta|^2 dx.\end{aligned}$$

Input this into (2.4), we can conclude that

$$\begin{aligned}&\int_\Sigma \left[v^2\eta^2 + u^{p+1}\eta^2\right]dx\\ &\le C_1\int_\Sigma u^2\Big[|\nabla(\Delta\eta)\cdot\nabla\eta| + (\Delta\eta)^2 + \big|\Delta(|\nabla\eta|^2)\big|\Big]dx + C_2\int_\Sigma uv|\nabla\eta|^2 dx.\end{aligned}\tag{2.5}$$

Take $\eta = \varphi^m$ with $m > 2$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \ge 0$, it follows that

$$\begin{aligned}\int_\Sigma uv|\nabla\eta|^2 dx &= m^2\int_\Sigma uv\varphi^{2(m-1)}|\nabla\varphi|^2 dx\\ &\le \frac{1}{2C}\int_\Sigma (v\varphi^m)^2 dx + C\int_\Sigma u^2\varphi^{2(m-2)}|\nabla\varphi|^4 dx.\end{aligned}$$

Choose now φ_0 a cut-off function in $C_0^\infty(B_2)$ verifying $0 \le \varphi_0 \le 1$, $\varphi_0 = 1$ for $|x| < 1$. Input the above inequality into (2.5) with $\varphi = \varphi_0(R^{-1}x)$ for $R > 0$, $\eta = \varphi^m$ and $m = \frac{2p+2}{p-1} > 2$, we

arrive at

$$\begin{aligned}\int_\Sigma \left(v^2+u^{p+1}\right)\varphi^{2m}dx &\le \frac{C}{R^4}\int_\Sigma u^2\varphi^{2m-4}dx\\ &\le \frac{C}{R^4}\left(\int_\Sigma u^{p+1}\varphi^{(p+1)(m-2)}dx\right)^{\frac{2}{p+1}}R^{\frac{N(p-1)}{p+1}}\\ &= \frac{C}{R^4}\left(\int_\Sigma u^{p+1}\varphi^{2m}dx\right)^{\frac{2}{p+1}}R^{\frac{N(p-1)}{p+1}}.\end{aligned}\tag{2.6}$$

Hence

$$\int_\Sigma u^{p+1}\varphi^{2m}dx \le CR^{N-\frac{4(p+1)}{p-1}}.$$

Combining with (2.6), as $\varphi^{2m}=1$ for $x\in B_R:=\{x\in\mathbb{R}^N,\ |x|\le R\}$, (2.1) is proved. □

Proof of Proposition 1.2. Let

$$\zeta=\beta u^{\frac{p+1}{2}}-v,\quad \text{where}\ \ \beta=\sqrt{\frac{2}{p+1}}.$$

Then a direct computation shows that $\Delta\zeta\ge\beta^{-1}u^{\frac{p-1}{2}}\zeta$ in Σ. Consider $\zeta_+:=\max(\zeta,0)$, there holds, for any $R>0$

$$\int_{\Sigma\cap B_R}|\nabla\zeta_+|^2dx=-\int_{\Sigma\cap B_R}\zeta_+\Delta\zeta dx+\int_{\partial(\Sigma\cap B_R)}\zeta_+\frac{\partial\zeta}{\partial\nu}d\sigma\le\int_{\Sigma\cap\partial B_R}\zeta_+\frac{\partial\zeta}{\partial\nu}d\sigma.\tag{2.7}$$

Here we used $\zeta_+\Delta\zeta\ge0$ in Σ and $\zeta=0$ on $\partial\Sigma$. Denote now S^{N-1} the unit sphere in $\mathbb{R}^N$ and

$$e(r)=\int_{S^{N-1}\cap(r^{-1}\Sigma)}\zeta_+^2(r\sigma)d\sigma\quad\text{for } r>0.$$

Remark that $\exists\,R_0>0$ such that

$$\int_{\Sigma\cap\partial B_r}\zeta_+\frac{\partial\zeta}{\partial\nu}d\sigma=\frac{r^{N-1}}{2}e'(r),\quad\forall\,r\ge R_0.\tag{2.8}$$

Moreover, for $R\ge R_0$, we deduce from (2.1)

$$\int_{R_0}^R r^{N-1}e(r)dr\le\int_{B_R\cap\Sigma}\zeta_+^2dx\le C\int_{B_R\cap\Sigma}\left(v^2+u^{p+1}\right)dx\le CR^{N-4-\frac{8}{p-1}}=o\left(R^N\right).$$

This means that the function e cannot be nondecreasing at infinity, so that there exists $R_j\to\infty$ satisfying $e'(R_j)\le0$. Combining with (2.7) and (2.8) with $R=R_j\to\infty$, there holds

$$\int_\Sigma|\nabla\zeta_+|^2dx=0.$$

Using $\zeta=0$ on $\partial\Sigma$, we have $\zeta_+\equiv0$ in Σ, or equivalently (1.8) holds true in Σ. Clearly $v>0$ in Σ by (1.8). □

In the following, we show some properties of the polynomials L and H, useful for our proofs. Let

$$L(s)=s^4-32\frac{p}{p+1}s^2+32\frac{p(p+3)}{(p+1)^2}s-64\frac{p}{(p+1)^2},\quad s\in\mathbb{R}.\tag{2.9}$$

Lemma 2.2. *$L(2t_0)<0$ and L has a unique root s_0 in the interval $(2t_0,\infty)$.*

Proof. Obviously

$$L(2t_0) = 16t_0^4 - 128\frac{p}{p+1}t_0^2 + 64\frac{p(p+3)}{(p+1)^2}t_0 - 64\frac{p}{(p+1)^2}$$

By $\frac{t_0^2}{2t_0-1} = \sqrt{\frac{2p}{p+1}}$ (see [2]), there holds $t_0^4 = \frac{2p}{p+1}(2t_0-1)^2$. A direct computation yields

$$\begin{aligned}\frac{(p+1)^2L(2t_0)}{32p} &= (p+1)(2t_0-1)^2 - 4(p+1)t_0^2 + 2(p+3)t_0 - 2\\ &= (p-1)(1-2t_0).\end{aligned}$$

As $t_0 > 1$ for any $p > 1$, we have $L(2t_0) < 0$. Furthermore, $\forall\, p > 1$, $s \geq 2t_0$, we have

$$\begin{aligned}(p+1)L''(s) = 12(p+1)s^2 - 64p &\geq 48(p+1)t_0^2 - 64p\\ &\geq 48(p+1)\frac{2p}{p+1} - 64p\\ &= 32p > 0\end{aligned}$$

in $[2t_0, \infty)$, where we used $t_0^2 \geq \frac{2p}{p+1}$ by (1.6). Therefore L is convex in $[2t_0, \infty)$, as $\lim_{s\to\infty} L(s) = \infty$ and $L(2t_0) < 0$, it's clear that L admits a unique root in $(2t_0, \infty)$. □

Remark 2.3. Performing the change of variable $x = \frac{p+1}{p-1}s$, a direct calculation gives

$$H(x) = \left(\frac{p+1}{p-1}\right)^4 L(s), \quad \text{hence } H(x) < 0 \text{ if and only if } L(s) < 0.$$

Using the above Lemma, $x_0 = \frac{p+1}{p-1}s_0$ is the largest root of the polynomial H, and x_0 is the unique root of H for $x \geq \frac{2(p+1)}{p-1}t_0$.

Lemma 2.4. *Let $x_0 = \frac{p+1}{p-1}s_0$ be the largest root of H. Then $x_0 > 5$ for any $p > 1$.*

Proof. As x_0 is the largest root of H, to have $x_0 > 5$, it is sufficient to show $H(5) < 0$. Let $J(p) = (p-1)^4H(5)$, then $J(p) = -15p^4 - 1284p^3 + 4262p^2 - 3844p + 625$. Therefore,

$$J'(p) = -60p^3 - 3852p^2 + 8524p - 3844, \quad J''(p) = -180p^2 - 7704p + 8524.$$

We see that $J'' < 0$ in $[2, \infty)$. Consequently $J'(p) < 0$ and $J(p) < 0$ for $p \geq 2$. Hence $x_0 > 5$ if $p \geq 2$. For $p \in (1, 2)$, there holds $x_0 > \frac{2(p+1)}{p-1}t_0 \geq 6t_0 > 5$ as $t_0 > 1$. □

3. Proof of Theorems 1.1 and 1.3

We will prove only Theorem 1.1, since the proof of Theorem 1.3 is completely similar, where we can change just B_r by $B_r \cap \mathbb{R}^N_+$.

The following result generalizes Lemma 4 in [2], which is a crucial argument for our proof. As above, the constant C always denotes a positive number which may change term by term, but does not depend on the solution u. For $k \in \mathbb{N}$, let $R_k := 2^kR$ with $R > 0$.

Lemma 3.1. *Assume that u is a classical stable solution of (1.1). Then for all $2 \leq s < s_0$, there is $C < \infty$ such that*

$$\int_{B_{R_k}} u^p v^{s-1} dx \leq \frac{C}{R^2}\int_{B_{R_{k+1}}} v^s dx, \quad \forall\, R > 0. \tag{3.1}$$

Proof. Let u be a classical stable solution of (1.1). Let $\phi \in C_0^2(\mathbb{R}^N)$ and $\varphi = u^{\frac{q+1}{2}}\phi$ with $q \geq 1$. Take φ into the stability inequality (1.11), we obtain

$$\sqrt{p}\int_{\mathbb{R}^N} u^{\frac{p-1}{2}} u^{q+1}\phi^2 \leq \int_{\mathbb{R}^N} u^{q+1}|\nabla\phi|^2 + \int_{\mathbb{R}^N} |\nabla u^{\frac{q+1}{2}}|^2\phi^2 + (q+1)\int_{\mathbb{R}^N} u^q\phi\nabla u\nabla\phi \tag{3.2}$$

Integrating by parts, we get

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u^{\frac{q+1}{2}}|^2\phi^2 dx &= \frac{(q+1)^2}{4}\int_{\mathbb{R}^N} u^{q-1}|\nabla u|^2\phi^2 dx\\ &= \frac{(q+1)^2}{4q}\int_{\mathbb{R}^N} \phi^2\nabla(u^q)\nabla u dx\\ &= \frac{(q+1)^2}{4q}\int_{\mathbb{R}^N} u^q v\phi^2 dx - \frac{q+1}{4q}\int_{\mathbb{R}^N} \nabla(u^{q+1})\nabla(\phi^2)dx\\ &= \frac{(q+1)^2}{4q}\int_{\mathbb{R}^N} u^q v\phi^2 dx + \frac{q+1}{4q}\int_{\mathbb{R}^N} u^{q+1}\Delta(\phi^2)dx\end{aligned} \tag{3.3}$$

and

$$(q+1)\int_{\mathbb{R}^N} u^q\phi\nabla u\nabla\phi dx = \frac{1}{2}\int_{\mathbb{R}^N} \nabla(u^{q+1})\nabla(\phi^2)dx = -\frac{1}{2}\int_{\mathbb{R}^N} u^{q+1}\Delta(\phi^2)dx. \tag{3.4}$$

Combining (3.2)-(3.4), we conclude that

$$a_1\int_{\mathbb{R}^N} u^{\frac{p-1}{2}} u^{q+1}\phi^2 dx \leq \int_{\mathbb{R}^N} u^q v\phi^2 dx + C\int_{\mathbb{R}^N} u^{q+1}\left(|\Delta(\phi^2)| + |\nabla\phi|^2\right)dx \tag{3.5}$$

where $a_1 = \frac{4q\sqrt{p}}{(q+1)^2}$. Choose now $\phi(x) = h(R_k^{-1}x)$ where $h \in C_0^\infty(B_2)$ such that $h \equiv 1$ in B_1, there holds then

$$\int_{\mathbb{R}^N} u^{\frac{p-1}{2}} u^{q+1}\phi^2 dx \leq \frac{1}{a_1}\int_{\mathbb{R}^N} u^q v\phi^2 dx + \frac{C}{R^2}\int_{B_{R_{k+1}}} u^{q+1}dx \tag{3.6}$$

Now, apply the stability inequality (1.11) with $\varphi = v^{\frac{r+1}{2}}\phi$, $r \geq 1$, there holds

$$\sqrt{p}\int_{\mathbb{R}^N} u^{\frac{p-1}{2}} v^{r+1}\phi^2 \leq \int_{\mathbb{R}^N} v^{r+1}|\nabla\phi|^2 + \int_{\mathbb{R}^N} |\nabla v^{\frac{r+1}{2}}|^2\phi^2 + (r+1)\int_{\mathbb{R}^N} v^r\phi\nabla v\nabla\phi$$

By very similar computation as above (recalling that $-\Delta v = u^p$), we have

$$\int_{\mathbb{R}^N} u^{\frac{p-1}{2}} v^{r+1}\phi^2 dx \leq \frac{1}{a_2}\int_{\mathbb{R}^N} u^p v^r\phi^2 dx + \frac{C}{R^2}\int_{B_{R_{k+1}}} v^{r+1}dx \tag{3.7}$$

where $a_2 = \frac{4r\sqrt{p}}{(r+1)^2}$.

Using (3.6) and (3.7), there holds

$$\begin{aligned}I_1 + a_2^{r+1}I_2 &:= \int_{\mathbb{R}^N} u^{\frac{p-1}{2}} u^{q+1}\phi^2 dx + a_2^{r+1}\int_{\mathbb{R}^N} u^{\frac{p-1}{2}} v^{r+1}\phi^2 dx\\ &\leq \frac{1}{a_1}\int_{\mathbb{R}^N} u^q v\phi^2 dx + a_2^r\int_{\mathbb{R}^N} u^p v^r\phi^2 dx + \frac{C}{R^2}\int_{B_{R_{k+1}}} \left(u^{q+1} + v^{r+1}\right)dx.\end{aligned} \tag{3.8}$$

Fix now

$$2q = (p+1)r + p - 1, \quad \text{or equivalently} \quad q + 1 = \frac{(p+1)(r+1)}{2}. \tag{3.9}$$

By Young's inequality, we get

$$\begin{aligned}\frac{1}{a_1}\int_{\mathbb{R}^N} u^q v\phi^2 dx &= \frac{1}{a_1}\int_{\mathbb{R}^N} u^{\frac{p-1}{2}} u^{\frac{p+1}{2}r} v\phi^2 dx\\ &= \frac{1}{a_1}\int_{\mathbb{R}^N} u^{\frac{p-1}{2}} u^{\frac{r}{r+1}(q+1)} v\phi^2 dx\\ &\le \frac{r}{r+1}\int_{\mathbb{R}^N} u^{\frac{p-1}{2}} u^{q+1}\phi^2 dx + \frac{1}{a_1^{r+1}(r+1)}\int_{\mathbb{R}^N} u^{\frac{p-1}{2}} v^{r+1}\phi^2 dx\\ &= \frac{r}{r+1}I_1 + \frac{1}{a_1^{r+1}(r+1)}I_2\end{aligned}$$

and similarly

$$a_2^r\int_{\mathbb{R}^N} u^p v^r\phi^2 dx \le \frac{1}{r+1}I_1 + \frac{a_2^{r+1}r}{r+1}I_2$$

Combining the above two inequalities and (3.8), we deduce then

$$a_2^{r+1}I_2 \le \left[\frac{a_2^{r+1}r}{r+1} + \frac{1}{a_1^{r+1}(r+1)}\right] I_2 + \frac{C}{R^2}\int_{B_{R_{k+1}}}\left(u^{q+1}+v^{r+1}\right)dx,$$

hence

$$\frac{(a_1a_2)^{r+1}-1}{r+1}I_2 \le \frac{Ca_1^{r+1}}{R^2}\int_{B_{R_{k+1}}}\left(u^{q+1}+v^{r+1}\right)dx.$$

Thus, if $a_1a_2 > 1$, by the choice of ϕ,

$$\int_{B_{R_k}} u^{\frac{p-1}{2}} v^{r+1} dx \le I_2 \le \frac{C}{R^2}\int_{B_{R_{k+1}}}\left(u^{q+1}+v^{r+1}\right)dx.$$

From (1.8) and (3.9), we get $u^{q+1} \le Cv^{r+1}$. Denote $s = r+1$, we can conclude that if $a_1a_2 > 1$,

$$\int_{B_{R_k}} u^p v^{s-1} dx \le C_1\int_{B_{R_k}} u^{\frac{p-1}{2}} v^s dx \le \frac{C_2}{R^2}\int_{B_{R_{k+1}}}\left(u^{q+1}+v^{r+1}\right)dx \le \frac{C_3}{R^2}\int_{B_{R_{k+1}}} v^s dx. \tag{3.10}$$

On the other hand, a simple verification shows that

$$a_1a_2 > 1 \text{ is equivalent to } L(s) < 0.$$

By Lemma 2.2, for $s \in [2t_0, s_0)$, there holds $L(s) < 0$. So the inequality (3.10), i.e. (3.1) holds true for any $2t_0 \le s < s_0$. On the other hand, by Lemma 4 of [2], the estimate (3.1) is valid for $2 \le s < 2t_0$, hence for $2 \le s < s_0$. □

Now, we can follow exactly the iteration process in [2] (see Proposition 1 or Corollary 2 there) to obtain

Corollary 3.2. *Suppose u is a classical stable solution of (1.1). For all $2 \le \beta < \frac{N}{N-2}s_0$, there are $\ell \in \mathbb{N}$ and $C < \infty$ such that*

$$\left(\int_{B_R} v^\beta dx\right)^{\frac{1}{\beta}} \le CR^{\frac{N}{2}(\frac{2}{\beta}-1)}\left(\int_{B_{R_{3\ell}}} v^2 dx\right)^{\frac{1}{2}}, \quad \forall\, R > 0.$$

Now we are in position to complete the proof of Theorem 1.1. Let u be a smooth stable solution to (1.1), combining Corollary 3.2 and (2.1), for any $2 \leq \beta < \frac{N}{N-2}s_0$, there exists $C > 0$ such that

$$\left(\int_{B_R} v^\beta dx\right)^{\frac{1}{\beta}} \leq CR^{\frac{N}{2}(\frac{2}{\beta}-1)+\frac{N}{2}-2-\frac{4}{p-1}}, \quad \forall\, R > 0.$$

Note that

$$\frac{N}{2}\left(\frac{2}{\beta}-1\right)+\frac{N}{2}-2-\frac{4}{p-1} < 0 \;\Leftrightarrow\; N < \frac{2(p+1)}{p-1}\beta.$$

Considering the allowable range of β given in Corollary 3.2, if $N < 2 + \frac{2(p+1)}{p-1}s_0$, after sending $R \to \infty$ we get then $\|v\|_{L^\beta(\mathbb{R}^N)} = 0$, which is impossible since v is positive. To conclude, the equation (1.1) has no classical stable solution if $N < 2 + 2x_0$ where $x_0 = \frac{p+1}{p-1}s_0$.

Moreover, by Lemma 2.4, $x_0 > 5$ for any $p > 1$, which means that if $N \leq 12$, (1.1) has no classical stable solution for all $p > 1$. □

4. Proof of Theorem 1.4

In this section, we consider the elliptic problem (P_λ). Let u_λ be the minimal solution of (P_λ), it is well known that u_λ is stable. To simplify the presentation, we erase the index λ. By [4, 5], there holds

$$\sqrt{\lambda p}\int_\Omega (u+1)^{\frac{p-1}{2}}\varphi^2 dx \leq \int_\Omega |\nabla\varphi|^2 dx, \quad \forall\, \varphi \in H_0^1(\Omega) \tag{4.1}$$

Using $\varphi = u^{\frac{q+1}{2}}$ as test function in (3.2), by similar computation as for (3.5) in section 3, we obtain

$$a_1\sqrt{\lambda}\int_\Omega (u+1)^{\frac{p-1}{2}}u^{q+1}dx \leq \int_\Omega u^q v dx, \quad \text{where } \; a_1 = \frac{4q\sqrt{p}}{(q+1)^2}. \tag{4.2}$$

Here we need not a cut-off function ϕ, because all boundary terms appearing in the integrations by parts vanish under the Navier boundary conditions, hence the calculations are even easier. We can use the Young's inequality as for Theorem 1.1, but we show here a proof inspired by [6].

Similarly as for (3.7), using $\varphi = v^{\frac{r+1}{2}}$ in (4.1), we have

$$a_2\sqrt{\lambda}\int_\Omega (u+1)^{\frac{p-1}{2}}v^{r+1}dx \leq \int_\Omega \lambda(u+1)^p v^r dx, \quad \text{where } \; a_2 = \frac{4r\sqrt{p}}{(r+1)^2}. \tag{4.3}$$

Take always $2q = (p+1)r + p - 1$. Applying Holder's inequality, there hold

$$\begin{aligned}\int_\Omega u^q v dx &\leq \left(\int_\Omega u^{\frac{p-1}{2}}v^{r+1}dx\right)^{\frac{1}{r+1}}\left(\int_\Omega u^{\frac{p-1}{2}+q+1}dx\right)^{\frac{r}{r+1}}\\ &\leq \left[\int_\Omega (u+1)^{\frac{p-1}{2}}v^{r+1}dx\right]^{\frac{1}{r+1}}\left(\int_\Omega u^{\frac{p-1}{2}+q+1}dx\right)^{\frac{r}{r+1}}\end{aligned} \tag{4.4}$$

and

$$\int_\Omega (u+1)^p v^r dx \leq \left[\int_\Omega (u+1)^{\frac{p-1}{2}}v^{r+1}dx\right]^{\frac{r}{r+1}}\left[\int_\Omega (u+1)^{\frac{p-1}{2}+q+1}dx\right]^{\frac{1}{r+1}}. \tag{4.5}$$

Multiplying (4.2) with (4.3), using (4.4) and (4.5), we get immediately

$$\left[\int_\Omega (u+1)^{\frac{p-1}{2}} u^{q+1} dx\right]^{\frac{1}{r+1}} \leq \frac{1}{a_1 a_2} \left[\int_\Omega (u+1)^{\frac{p-1}{2}+q+1} dx\right]^{\frac{1}{r+1}}. \tag{4.6}$$

On the other hand, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$(u+1)^{\frac{p-1}{2}+q+1} \leq (1+\varepsilon)(u+1)^{\frac{p-1}{2}} u^{q+1} + C_\varepsilon \quad \text{in } \mathbb{R}_+.$$

If $a_1 a_2 > 1$, there exists $\varepsilon_0 > 0$ satisfying $1 + \varepsilon_0 < (a_1 a_2)^{r+1}$. We deduce from (4.6) that

$$\left[1 - \frac{1+\varepsilon_0}{(a_1 a_2)^{r+1}}\right] \int_\Omega (u+1)^{\frac{p-1}{2}} u^{q+1} dx \leq C.$$

Therefore, when $L(s) < 0$, i.e. when $a_1 a_2 > 1$, there is $C > 0$ such that

$$\int_\Omega u^{\frac{p-1}{2}+q+1} dx \leq \int_\Omega (u+1)^{\frac{p-1}{2}} u^{q+1} dx \leq C.$$

As $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$, we conclude, using Lemma 2.2,

$$u^* \in L^{\frac{p-1}{2}+q+1}(\Omega), \quad \text{for all } q \text{ satisifying } \frac{2(q+1)}{p+1} = r + 1 = s < s_0. \tag{4.7}$$

Furthermore, by [10], we know that $u^* \in H^2(\Omega)$. As $u^* \geq 0$ verifies $\Delta^2 u^* = \lambda^*(u^*+1)^p \leq C(u^*)^{p-1}u^* + C$ with $u^* = \Delta u^* = 0$ on $\partial\Omega$, by standard elliptic estimate, we know that u^* is smooth if

$$\frac{N}{4} < \left(\frac{p-1}{2} + q + 1\right)\frac{1}{p-1} = \frac{1}{2}\left(1 + \frac{p+1}{p-1}s\right).$$

Therefore, u^* is smooth if $N < 2+2x_0$. By Lemma 2.4, u^* is smooth for any $p > 1$ if $N \leq 12$. □

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